MATHEMATICS

RATIONAL QUADRILATERALS

BY

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§ 1. *Introduction*

A polygon, the sides and diagonals of which have a rational length, will be called a rational polygon. The problem of finding rational quadrilaterals has interested mathematicians since antiquity. The first who supplied a method to find *all* possible rational quadrilaterals was KUMMER in 1848. (See [1], p. 48–50.) The disadvantage of his method is that quadrilaterals with comparatively small integer sides and diagonals are only found by giving complicated values to the parameters in his formulae. In 1950 the author (see [1], p. 42–48) published a method, based on the idea of the "index" of a rational triangle, which also gives all possible rational quadrilaterals, without the above mentioned disadvantage.

Recently investigations have established, that there are an infinity of rational quadrilaterals. At the end of 1959 A. S. BESICOVITCH [2] published a proof of the following theorem: The class of rational parallelograms is everywhere dense in the class of all parallelograms, i.e. given any parallelogram there exist rational parallelograms whose sides and diagonals have lengths that differ arbitrarily little from the corresponding lengths of the given parallelogram. Also in 1959 L. J. MORDELL [3] gave a proof of a more general theorem which states: Given any quadrilateral $A'B'C'D'$, there exist rational quadrilaterals $ABCD$ whose vertices A, B, C, D are close (i.e. arbitrarily near) A', B', C', D'.

In this paper we will give a proof of a further generalisation:

Theorem. Given three distinct points A, B, C with rational distances AB, BC, CA; the collection of points P for which PA, PB, PC are rational, is everywhere dense in any plane that contains A, B, C.

The reader will be able to derive without difficulty Mordell's theorem from ours.

Remark. The assumption that the lengths of AB, BC, CA be rational is necessary, as appears from the following example. If $AB=BC=2$ and $AC=\pi$, then for every point P with rational AP, BP, CP it appears that AC is equal to a root of a fourth degree equation with integer coefficients, which contradicts $AC=\pi$.

§ 2. *The index of a rational triangle*

The area of a rational triangle has always a unique representation $r\sqrt{\omega}$ where r is a rational number and ω a "squarefree" natural number. This number ω will be called the "index" of the rational triangle.

Let ABC be a rational triangle with index ω and angles equal to α, β and γ. Then

$$\tan \tfrac{1}{2}\alpha = x\sqrt{\omega},\ \tan \tfrac{1}{2}\beta = y\sqrt{\omega},$$

where x and y are positive rational numbers with

$$xy < 1/\omega.$$

It is easily seen that the diagonals of a rational quadrilateral divide each other into rational segments and that the triangles, into which a rational quadrilateral is divided by the diagonals, all have the same index. Let $ACBE$ be a rational quadrilateral with the diagonals AB and CE which intersect at the point D. Let α, β and γ be the angles of the triangle ABC. We now put:

$$\tan \tfrac{1}{2}\alpha = x_0\sqrt{\omega};\ \tan \tfrac{1}{2}\beta = y_0\sqrt{\omega};\ \tan \tfrac{1}{2}CDB = u_0\sqrt{\omega}$$

where $x_0, y_0,\ u_0$ are positive rational numbers satisfying the relation

$$x_0\, y_0 < 1/\omega.$$

If AD/DB is equal to m/n, where m and n are integers, the following relation in m and n holds (see [1] p. 42):

$$\text{(1)}\qquad n\left(\frac{1}{x_0} - \omega\, x_0\right) + m\left(\omega\, y_0 - \frac{1}{y_0}\right) + (m+n)\left(\omega\, u_0 - \frac{1}{u_0}\right) = 0.$$

We now put

$$x_0 = x_1/x_2,\ y_0 = y_1/y_2,\ u_0 = u_1/u_2,$$

where the separate pairs $x_1, x_2;\ y_1, y_2;\ z_1, z_2$ are relatively prime positive integers.

For m and n, the ratio of which was determined by (1), we take the integer values

$$\text{(2)}\qquad \begin{cases} m = y_1\, y_2(x_2\, u_1 - x_1\, u_2)\ (\omega\, x_1\, u_1 + x_2\, u_2) \\ n = x_1\, x_2(y_2\, u_1 + y_1\, u_2)\ (-\omega\, y_1\, u_1 + y_2\, u_2) \end{cases}$$

from which we obtain

$$\text{(2A)}\quad m - n = 2\, x_1\, x_2\, y_1\, y_2(\omega\, u_1{}^2 - u_2{}^2) + (x_2\, y_1 - x_1\, y_2)\ (\omega\, x_1\, y_1 + x_2\, y_2) u_1\, u_2.$$

Next we consider the triangle ABE with the angles α' and β' at A and B. Then

$$\tan \tfrac{1}{2}\alpha' = x'\sqrt{\omega},\ \tan \tfrac{1}{2}\beta' = y'\sqrt{\omega},$$

where x' and y' are positive rational numbers.

If C and E are on the same side of AB, then x' and y' will be connected by a relation which can be derived from (1) by replacing x by x' and y by y'. If however C and E are on different sides of AB then x' and y' will satisfy a relation that follows from (1) by replacing x_0, y_0 and u_0 by x', y' and $-u_0$ respectively.

§ 3. *The cubic curve*

Let ABC be a given rational triangle with $AC \neq BC$. We take a point D on AB which divides AB into a given rational proportion. The numbers $x_1, x_2, y_1, y_2, u_1, u_2$ and consequently m and n are then determined. The equations

$$(3) \qquad n\left(\frac{1}{x} - \omega x\right) + m\left(\omega y - \frac{1}{y}\right) + (m+n)\left(\omega u - \frac{1}{u}\right) = 0$$

$$(4) \qquad n\left(\frac{1}{x} - \omega x\right) + m\left(\omega y - \frac{1}{y}\right) - (m+n)\left(\omega u - \frac{1}{u}\right) = 0$$

represent cubic curves in the coordinates x and y. Any rational point (x, y) (i.e. a point with rational coordinates) on these curves, different from (x_0, y_0), and with $x>0$, $y>0$ and $xy<1/\omega$, yields a rational quadrilateral. By changing the signs of x and y (4) is transformed into (3). Thus it is obviously sufficient to consider (3) only. The curve (3) has the rational points

$$(u, -u); \left(u, \frac{1}{\omega u}\right); \left(-\frac{1}{\omega u}, -u\right); \left(-\frac{1}{\omega u}, \frac{1}{\omega u}\right).$$

Equation (3) can be written in the form

$$(5) \qquad ax + \frac{b}{x} + cy + \frac{d}{y} + e = 0.$$

By means of the birational transformation (see [1], p. 95):

$$x = \frac{-2b\lambda(Y - e\lambda X)}{X(X-4ab\lambda^2)}, \quad y = \frac{Y - e\lambda X}{2c\lambda(X-4ab\lambda^2)}$$

whose inverse is:

$$X = -4bc\lambda^2 \cdot \frac{y}{x}, \quad Y = -4bc\lambda^3 \cdot \frac{y}{x} \cdot (2ax + 2cy + e),$$

equation (5) is transformed into

$$Y^2 = X^3 + (e^2 - 4ab - 4cd)\lambda^2 X^2 + 16abcd\lambda^4 X.$$

Here λ is an arbitrary rational constant. By taking $\lambda = u_1u_2$ the coefficients of the transformed equation become *integers*. The form of the equation is then:

$$Y^2 = X^3 + \{(m+n)^2 (\omega u_1^2 - u_2^2)^2 + 4\omega u_1^2 u_2^2 (m^2+n^2)\}X^2 + 16m^2n^2\omega^2 u_1^4 u_2^4 X,$$

or

$$(6) \qquad Y^2 = X^3 + (N^2 - 2M)X^2 + M^2 X$$

where

$$M = 4mn\omega u_1^2 u_2^2,\ N = (m+n)(\omega u_1^2 + u_2^2).$$

The transformation formulae are

$$(7)\qquad \begin{cases} X = -4mn\omega u_1^2 u_2^2 \cdot \dfrac{y}{x} \\ Y = -4mn\omega u_1^3 u_2^3 \cdot \dfrac{y}{x} \bullet \left\{2\omega(my-nx) + (m+n)\left(\omega u - \dfrac{1}{u}\right)\right\} \end{cases}$$

Since m and n are integers, equation (6) has indeed integer coefficients. The cubic curve represented by this equation intersects the X-axis in three distinct points, one of which is the origin. In fact, the discriminant of the right-hand side of (6) has the same sign as $N^2 - 4M$ and

$$N^2 - 4M = (m+n)^2 (\omega u_1^2 + u_2^2)^2 - 16mn\omega u_1^2 u_2^2 \geqq 0$$

since

$$(m+n)^2 \geqq 4mn,\ (\omega u_1^2 + u_2^2)^2 \geqq 4\omega u_1^2 u_2^2.$$

The sign of equality appears only if $\omega u_1^2 - u_2^2 = 0$ and $m = n$. As appears from (2A) this leads to $\omega = 1$, $x_1 = y_1$, $x_2 = y_2$ and thus $AC = BC$, which has been excluded. The curve (6) consists of an oval which lies to the left of the Y-axis and an infinite branch all of whose points lie to the right of the Y-axis, with the exception of the intersection with the X-axis, which is the origin.

§ 4. *Exceptional points*

The rational points $(u, -u)$ and $(u, 1/\omega u)$ of the cubic (3) are transformed by (7) into points $P_0(X_0, Y_0)$ and $P_1(X_1, Y_1)$ with the abscissae:

$$X_0 = 4mn\omega u_1^2 u_2^2,\ X_1 = -4mnu_2^4$$

since for all m and n the product $X_0 X_1 < 0$, the oval has at least *one* rational point.

After some calculation we obtain

$$Y_1 = -4mnu_2^4 (m-n)(u_2^2 + \omega u_1^2).$$

Without loss of generality we may assume $m \neq n$ (see § 5). The tangent at the point P_1 intersects the curve at a point $P_2(X_2, Y_2)$, called the tangential point of P_1. In general the tangential point of a point (x, y) of the cubic

$$(8)\qquad y^2 = x^3 + Kx^2 + Lx$$

has the abscissa

$$x' = \left(\frac{x^2 - L}{2y}\right)^2.$$

If (x, y) is a rational point, the tangential point (x', y') also will be rational. One can proceed in this way and compute (x'', y''), the tangential point of (x', y'), etc. This method leads in general to an infinity of rational points, but sometimes the process gives only a finite number of rational points and then we call (x, y) an *exceptional* point.

In the present case we find

$$X_2 = \Theta^2, \text{ where } \Theta = \frac{2mn}{m-n} \cdot (\omega u_1^2 - u_2^2).$$

We now use a theorem by T. NAGELL ([4], p. 15), which states that an exceptional rational point of the cubic

$$\eta^2 = \xi^3 + A\xi + B \tag{9}$$

where A and B are integers, has integer coordinates.

By means of the transformation

$$x = \tfrac{1}{9}\xi - \tfrac{1}{3}K, \quad y = \tfrac{1}{27}\eta$$

equation (8) is transformed into an equation of the form (9); from this we deduce the theorem: If (x, y) is an exceptional rational point of the cubic (8), where K and L are integers, then the quantity $9x$ is an integer. The point P_2 of the curve (6) can be exceptional only if $9X_2$ is an integer. In that case 3Θ is an integer; hence if P_2 is an exceptional point,

$$\frac{6mn(\omega u_1^2 - u_2^2)}{m-n} \tag{10}$$

is an integer.

We consider the numerator and the denominator of this fraction as polynomials in u_1 and u_2.

It appears from (2) and (2A) that in these variables the numerator is homogeneous of the sixth degree and the denominator homogeneous of the second degree. Since ω is a square-free number, it is not possible to decompose $\omega u_1^2 - u_2^2$ into rational factors. It appears from (2A) that the denominator, considered as a function of u_1 and u_2 is divisible by $\omega u_1^2 - u_2^2$ only if $x_2 y_1 - x_1 y_2 = 0$. However the last relation does not hold since $AC \neq BC$. The denominator, considered as a function of u_1 and u_2 can have a factor in common with mn only if m and n, considered as functions of u_1 and u_2 have a factor in common. This too is impossible as appears upon a closer inspection of the expressions (2) in connection with the fact that x_1, x_2, y_1, y_2 represent *positive* integers which satisfy the relation $\omega x_1 y_1 < x_2 y_2$. Hence the numerator $f(u_1, u_2)$ and the denominator $g(u_1, u_2)$ of the fraction (10) are homogeneous polynomials in u_1 and u_2 with integer coefficients and without a common factor in u_1 and u_2. These polynomials f and g are uniquely determined, once the triangle ABC is given.

Since f and g are relatively prime, two homogeneous polynomials with integer coefficients $h(u_1, u_2)$ and $k(u_1, u_2)$ exist, such that:

$$f(u_1, u_2) \cdot h(u_1, u_2) + g(u_1, u_2) \cdot k(u_1, u_2) = R \tag{11}$$

where $R \neq 0$ is an integer. Now suppose that u_1 and u_2 assume such values, that f/g is *not* an integer. In that case the curve (6) has a non-exceptional rational point and consequently has an infinity of rational points. We use a well-known result of Poincaré, which states that if there is an infinity of rational points on a curve of the type (8), these points will be everywhere dense on the infinite branch; the same holds for the oval provided there is at least one rational point on the oval. Then in the case that f/g is not an integer, the curve (6) has an infinity of rational points, which are everywhere dense on the curve. But then the same holds for the curve (3), which may be obtained from (6) by a *continuous* rational transformation.

§ 5. *Approximation of an arbitrary quadrilateral by rational quadrilaterals*

Let ABC be a non-equilateral rational triangle. We may suppose that $AC \neq BC$. Let E be a point in the plane ABC; CE meets AB in a point D. Our aim is to prove that in any neighbourhood of E there is a point E' such that $ACBE'$ is a rational quadrilateral. Without loss of generality we may suppose that $AD \neq DB$, for if $AD = DB$ there exists a point E^* sufficiently close to E, for which the corresponding point D^* satisfies $AD^* \neq D^*B$, and if a point E' with the desired property can be found in a suitably chosen neighbourhood of E^*, this point is in the desired neighbourhood of E.

The positive integers $\omega, x_1, x_2, y_1, y_2$ are determined. Putting

$$\tan \tfrac{1}{2}BAE = x'\sqrt{\omega}, \ \tan \tfrac{1}{2}BDC = u\sqrt{\omega},$$

the positive numbers x' and u need not be rational in general. Let $u' = p/q$ be a rational number close to u, where p and q are relatively prime integers. By a suitable choice of p and q it is always possible to attain $f(p, q) \neq 0$ and $g(p, q) \neq 0$. Proceeding in a manner similar to the method used by Mordell [3], p. 281, we put

$$u_1 = pt + r, \ u_2 = qt + s,$$

where r and s are integers with the condition $ps - qr \neq 0$.

By this substitution $f(u_1, u_2)$ and $g(u_1, u_2)$ become polynomials $F(t)$ and $G(t)$ with integer coefficients. Equation (11) becomes:

$$F(t) \cdot \varphi(t) + G(t) \cdot \psi(t) = R$$

where $\varphi(t)$ and $\psi(t)$ are polynomials with integer coefficients.

If t is an integer, the values of F and G are integers as well. A common factor of the integers F and G is also a factor of . When t tends to infinity, $|F|$ and $|G|$ tend to infinity also. Since F and G can have only a finite number of factors in common, we conclude that an infinity of values t exists, for which $F(t)/G(t)$ is *not* an integer. Thus it is possible to choose t so large that u_1/u_2 is arbitrarily close to p/q and moreover $F(t)/G(t)$ is not an integer. The curve (3) corresponding to such a value of t has an infinity of rational points which are everywhere dense on the curve. So there are rational points on this curve, the abscissae of which are arbitrarily close to x'. In this manner we obtain rational quadrilaterals $ACBE'$, with E' arbitrarily close to E.

§ 6. *The equilateral case*

In the preceding pages we supposed triangle ABC to be non-equilateral. In the equilateral case $x_1=y_1=1$, $x_2=y_2=\omega=3$. Then equation (6) becomes:

$$(12)\quad \begin{cases} Y^2 = X^3+(18u_1u_2)^2\,(90u_1^4-36u_1^2u_2^2+10u_2^4)\,X^2+(18u_1u_2)^4 \\ \qquad\qquad \{3(9u_1^2-u_2^2)\,(u_1^2-u_2^2)\}\,X. \end{cases}$$

The abscissa of the point P_1 is

$$X_1 = 324(9u_1^2-u_2^2)\,(u_1^2-u_2^2)u_2^4$$

which is an integer. However in this case it is not necessary to compute the tangential point of P_1. For a transformation exists, by which the coefficients of the equation of the curve remain integers, whereas the abscissa of the point into which P_1 is transformed, will not be an integer in general.

In order to achieve this, we put

$$X = (18u_1u_2)^2\xi,\ \ Y = (18u_1u_2)^3\eta,$$

by which (12) becomes

$$\eta^2 = \xi^3+(90u_1^4-36u_1^2u_2^2+10u_2^4)\xi^2+9(9u_1^2-u_2^2)^2\,(u_1^2-u_2^2)^2\xi.$$

Then X_1 is transformed into

$$\xi_1 = \frac{u_2^2(u_1^2-u_2^2)\,(9u_1^2-u_2^2)}{u_1^2}.$$

The numerator and the denominator of this fraction are relatively prime polynomials in u_1 and u_2, and the above method can be applied.

§ 7. *The collinear case*

Let A, B, C be collinear points with rational distances. If E is a point on the line AB, then there are on the line AB points E' arbitrarily near E, for which $E'A$, $E'B$, $E'C$ are rational. Now suppose E is a point not on the line AB. We set the lengths of AB, BE, EA, EC equal to c, a, b, d respectively and $AC : CB = m : n$, where m and n are rational numbers with $m+n=1$. Then

$$(13)\qquad d^2 = ma^2+nb^2-mnc^2.$$

Putting further

$$c = ua + u'b, \tag{14}$$

we will show that it is always possible to take the rational numbers u and u' in such a way, that for every choice of the rational numbers a and b the number d is rational. By substituting (14) into (13) we find

$$d^2 = (1-nu^2)ma^2 + (1-mu'^2)nb^2 - 2mnuu'ab.$$

The right-hand side is a quadratic form in a and b, whose discriminant must be zero. This leads to

$$mu'^2 + nu^2 = 1. \tag{15}$$

This condition is not only necessary but also sufficient, for as a consequence of (15)

$$d^2 = (amu' - bnu)^2. \tag{16}$$

If the numbers a, b, c, d, m, n are given, related by the condition (13), it is always possible by means of the equations (14) and (16) to find values for u and u'. These will satisfy (15). Consequently belong to any point E values of u and u' (rational or irrational), satisfying (15). In this manner any point E yields a point P on a conic with the equation (15). This conic has the rational point (1, 1) and hence an infinite number of rational points, which are everywhere dense on this conic. Therefore this conic has rational points P' that are arbitrarily close to P. Such a point P' yields a point E', situated in the plane ABE, arbitrarily close to E, for which the line-segments AE', BE', CE' have a rational length. This completes the proof of the theorem mentioned at the end of § 1.

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